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# Probability Models I

## Exponential Distribution and Poisson Process

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- Making mathematical models for real-world situation
  - We must make enough simplifying assumptions to enable us to handle the mathematics, but not so many that the model no longer resembles the real-world phenomenon
  - One widely used assumption is that certain random variables are exponentially distributed
    - Easy to work
    - Good approximation to the actual situation
    - Does not deteriorate with time

# Exponential Distribution

- Definition: a continuous random variable X is said to have an *exponential distribution* with parameter $\lambda\ (\lambda > 0)$ , if its probability density function is given by $f(x) = \begin{cases} \lambda e^{-\lambda x}, x \geq 0 \\ 0, x < 0 \end{cases}$

- Cumulative distribution function (CDF)

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \begin{cases} 1 - e^{-\lambda x}, & x \geqslant 0 \\ 0, & x < 0 \end{cases}$$

- Mean $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx \ = \frac{1}{\lambda}$
- Variance

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 \ = \frac{1}{\lambda^2}$$

# Example

- Update to a database follows exponential distribution with rate λ, at time x, what is the probability that the database is not updated?

$$P(X > x) = 1 - P(X < x) = 1 - \int_{-\infty}^{x} \lambda e^{-\lambda t} dt =$$

$$1 - (-e^{\lambda t}|_{-\infty}^{x} = 1 - (-e^{-\lambda x} - (-1)) \; = e^{-\lambda x}$$

- Properties:
- (1)Memoryless

$$P\{X > s + t \mid X > t\} = P\{X > s\} \qquad \text{for all } s, t \geqslant 0$$

- (2) Distribution of the minimum of exponential random variables
  - Let $X_1, \cdots, X_n$ be independent exponential distribution with rate $\lambda_1, \cdots, \lambda_n$
  - Then $min\{X_1, \cdots, X_n\}$ is also exponentially distributed with rate $(\lambda_1 + \cdots + \lambda_n)$

$$\Pr(\min\{X_1, \ldots, X_n\} > x) = \Pr(X_1 > x \text{ and } \ldots \text{ and } X_n > x)$$
$$= \prod_{i=1}^{n} \Pr(X_i > x) = \prod_{i=1}^{n} \exp(-x\lambda_i) = \exp\left(-x \sum_{i=1}^{n} \lambda_i\right).$$

- (3) $P(X_1 < X_2) = ?$

$$
\begin{aligned}
P\{X_1 < X_2\} &= \int_0^\infty P\{X_1 < X_2 | X_1 = x\} \lambda_1 e^{-\lambda_1 x}\, dx \\
&= \int_0^\infty P\{x < X_2\} \lambda_1 e^{-\lambda_1 x}\, dx \\
&= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x}\, dx \\
&= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) x}\, dx \\
&= \frac{\lambda_1}{\lambda_1 + \lambda_2}
\end{aligned}
$$

  - The index of the variable which achieves the minimum is distributed according to the law

$$
\Pr(X_k = \min\{X_1, \dots, X_n\}) = \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_n}.
$$

# Example

- Customers are waiting in line to receive service from a server
- Each customer waits an exponentially distributed time (independently) with rate θ; if the service has not yet begun by this time, it will depart the system immediately
- The service time of each customer is independent exponential random variables with rate μ.
- Suppose someone is presently being served.
- Consider the *n*th customer in the line:
  - (1) find $P_n$, the probability that the customer is eventually served.
  - (2) find $W_n$, the conditional expected amount of waiting time given that the customer is eventually served.

- (1) $P_n$=?
- Consider the n+1 random variables: n exponentials with rate θ, and one with μ
  - If waiting time of the *n*th customer is the smallest, it will not be served
    - The probability is $\frac{\theta}{n\theta+\mu}$
  - Otherwise, the smallest one will depart first, and the conditional probability that this customer will be served is the same as if it is in position n-1
  - Thus $P_n = (1-\frac{\theta}{n\theta+\mu})P_{n-1}$
  - Deduction on it yields:

$$P_n = \frac{(n-1)\theta+\mu}{n\theta+\mu}\frac{(n-2)+\mu}{(n-1)\theta+\mu}P_{n-2} = \cdots = \frac{\mu}{n\theta+\mu}$$

- (2)$W_n$=?
- We use the fact that min(*) is exponentially distributed with rate equal to the sum of rates
- Given the nth customer will be served eventually, the waiting time is the minimal of the n+1 random variables plus the additional time thereafter
- Use the memoryless property, we have

$$W_n = \frac{1}{n\theta + \mu} + W_{n-1}$$

- Thus

$$W_n = \sum_{i=1}^{n} \frac{1}{i\theta + \mu}$$

# Poisson Process

- Definition: counting process
  - A stochastic process {N(t), t≥0}, N(t) represents the total number of "events" that occur by t.
  - Example
    - The number of customers who enter a store
    - The number of children born in a period of time
- Definition: independent increment
  - A counting process is said to process independent increments if the number of events that occur in disjoint time intervals are independent.
- Definition: stationary increment
  - A counting process is said to process stationary increments if the distribution of the number of events that occur in any interval of time depends only on the length of the time interval

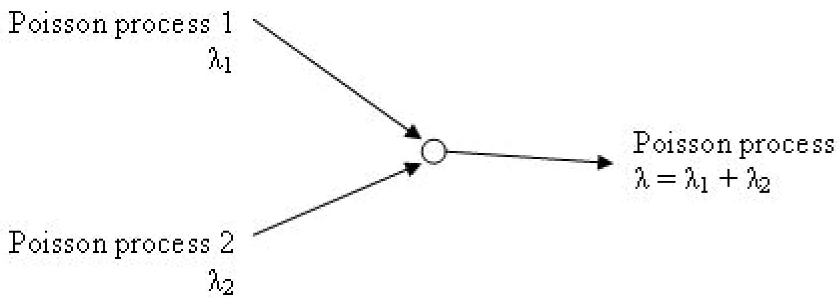
# Poisson Process

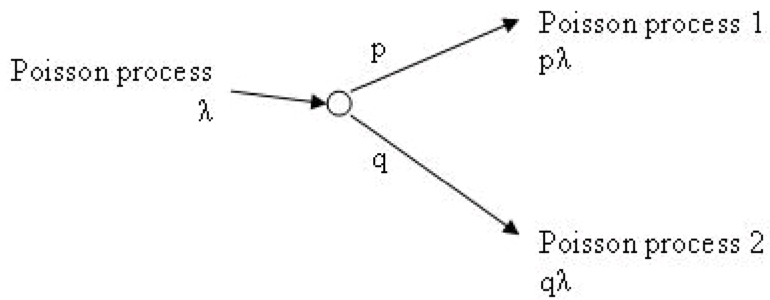
- Definition: Poisson process
- The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda$, $\lambda > 0$, if
- (i) $N(0)=0$
- (ii) The process has independent increments
- (iii) The number of events in any interval of length t is Poisson distributed with mean $\lambda t$. That is, for all s, t $\geq 0$

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \qquad n = 0, 1, \ldots$$

- (1) Poisson process has *stationary increments*, and $E[N(t)] = \lambda t$
- (2) Average time between events: $1/\lambda$
- (3) Average number of events in interval: (*Mean* of N(t):)   $E[N(t)]=\lambda t$
- (4)  *Memorylessness*
- (5) *Interarrival time exponential distributed*:
  - Let  $T_n$ (n=1,2, …) denote the elapsed time between the (n-1)st event and the nth event
  - Theorem: $T_n$, n=1,2,…, are independent identically distribution exponential random variables having mean $1/\lambda$.
- (6) *Events are uniformly distributed*
  - Theorem: given that N(t)=n, the n arrival times $S_1,\ldots,S_n$ have the same distribution as n independent random variables uniformly distributed on the interval (0,t).

- (7) *Superposition* and *Decomposition*
  - Superposition of two independent Poisson process with λ1 and λ2 yields a new Poisson process with λ= λ1+ λ2
  - If a Poisson process is split (decomposed) into two by selecting events for the first process with probability p and events for the second process with probability q = 1-p, then the two processes are independent Poisson processes with parameters p λ and q λ respectively.



- (8) *Mean* of decomposed Poisson process
  - Theorem: If $\{N_i(t); t \geq 0\}$, $i=1,..,k$ represent the number of type i events occurring in (0,t] and if $P_i(t)$ is the probability that an event occurring at time t is of type i, then $N_i(t)$ are independent Poisson random variables having means

$$E[N_i(t)] = \lambda \int_0^t P_i(s)\, ds$$

# An Optimization Example

- A server handles tasks in batch at a fixed time $T$
- Tasks arrive in accordance with Poisson process with rate $\lambda$
- A new server is added, which handles tasks in batch at an intermediate time $t \in (0,T)$
- Find the best $t$ to minimize the total expected waiting time of all tasks.

- Solution:
- The expected number of arrivals in (0,t) is λt
- Each arrival is uniformly distributed in (0,t), which has the expected waiting time t/2
- Thus the total expected waiting time of tasks arriving in (0,t) is λt*t/2
- Similar for arrivals in (t,T), which is λ(T-t)*(T-t)/2
- The expected waiting time of all tasks is $\frac{\lambda t^2}{2} + \frac{\lambda (T-t)^2}{2}$
- To achieve minimal, let $$\frac{d}{dt}\left[\lambda \frac{t^2}{2} + \lambda \frac{(T-t)^2}{2}\right] = \lambda t - \lambda (T-t) = 0$$
- Yields $t = T/2$

## Example: Tracking the Number of HIV Infections

- One of the difficulties in tracking the number of HIV infected people is its long incubation time, that is an infected person does not show any symptoms for a number of years, but is capable of infecting others.
- As a result, it is difficult for public health officials to be certain of the number of members of the population that are infected at any given time.
- We want to estimate the number of infected.

- Assumptions
  - Suppose that individuals contract HIV according to a *Poisson process* with unknown rate λ (we want to estimate λ)
  - Suppose that the time from when an individual becomes infected until symptoms of the disease appear is a random variable having *a known distribution* with cdf G
  - $N_1(t)$ be the number of individuals that have shown symptoms at time t
  - $N_2(t)$ be the number that have contracted HIV at time t but not yet shown symptoms
  - We have known the number of individuals with symptoms at time t is $n_1$

- Solution
  - An individual that contracts HIV at time s will show symptoms at time t with probability G(t-s), not show symptoms with probability 1-G(t-s)
  - According to *mean of decomposed Poisson process*

$$E[N_1(t)] = \lambda \int_0^t G(t-s)ds = \lambda \int_0^t G(x)dx$$

$$E[N_2(t)] = \lambda \int_0^t 1 - G(t-s)ds = \lambda \int_0^t 1 - G(x)dx$$

  - Let $E[N_1(t)] = \lambda \int_0^t G(x)dx \simeq n_1$ , thus $\lambda \simeq \frac{n_1}{\int_0^t G(x)dx}$
  - So

$$n_2 \simeq E[N_2(t)] = \lambda \int_0^t 1 - G(x)dx \simeq n_1 \frac{\int_0^t 1-G(x)dx}{\int_0^t G(x)dx}$$

  - For example, if G is exponential distribution with rate μ,

$$n_2 \simeq n_1 \frac{\mu\left(1-e^{-t/\mu}\right)}{t-\mu(1-e^{-t/\mu})}$$

    - E.g, if t=16 years, μ=10 years and $n_1$=220,000, then $n_2$=219,00

# Coupon Collector's Problem

- There are m different types of coupons
- Each time a person collect a coupon, he get the type j ($1 \leq j \leq m$) coupon with probability $p_j$, $(\sum_{j=1}^{m} p_j = 1)$
- Let N denote the number of coupons one needs to collect in order to have a complete collection of at least one of each type.
- Find E(N)=?

- We can solve the problem use *Poisson process*
- Suppose that coupons are collected at times chosen according to a *Poisson process with rate* $\lambda=1$.
- Say that an *event* is of type j, $1 \le j \le m$, if the coupon obtained is a type j coupon.
- Let $N_j(t)$ denote the number of type j coupons collected by time t
- According to *decomposition*, $\{N_j(t), t \ge 0\}$, $j=1,\ldots,m$, are independent Poisson process with rate $\lambda p_j = p_j$

- Let $X_j$ denote the time of the first event of the *j*th process
- Then $X = \max_{1 \leq j \leq m} X_j$ denote the time at which a complete collection is massed
- Since $X_j$ are independent, exponentially distributed with $p_j$

$$
\begin{aligned}
P\{X < t\} &= P\{\max_{1 \leqslant j \leqslant m} X_j < t\} \\
&= P\{X_j < t, \text{ for } j = 1, \ldots, m\} \\
&= \prod_{j=1}^{m} (1 - e^{-p_j t})
\end{aligned}
$$

- Therefore,

$$E[X] = \int_0^\infty x f_X(x)dx = \int_0^\infty \int_0^x dt f_X(x)dx =$$

$$\int_0^\infty \int_x^\infty f_X(x)dxdt = \int_0^\infty P(X > t)dt = \int_0^\infty \left[ 1 - \prod_{j=1}^{m}(1 - e^{-p_j t}) \right] dt$$

- Now link E(X) to E(N)
  - E(X): the expected time until one has a complete set
  - E(N): the expected number of coupons it takes
  - Let $T_i$ be the $i^{th}$ interarrival time finding the $(i-1)^{st}$ and the $i^{th}$ coupon, then $X=\sum T_i$
  - Since $T_i \sim Exp(1)$, and they are independent,

    $E[X|N]=E[\sum T_i|N]=N*E[T_1|N]=N$
  - Therefore, $E[X]=E\{E[X|N]\}=E[N]$

# Example: Neighbor Discovery in Wireless Networks (Mobicom 2009)

- Neighbor discovery
  - Immediately after deployment of wireless devices, a node have knowledge of other node in its communication range
  - It needs to discover its neighboring node to form the communication network
- Network model
  - Unique Node IDs
  - Radio model: Each node is equipped with a radio transceiver that allows a node to either transmit or receive messages, but not simultaneously.
  - Collision model: When two or more nodes transmit concurrently, a collision occurs at the recipient node, and no packets is received.

- Assumptions
  - 1. Consider a single clique of size n.
  - 2. n is known to all nodes in the clique.
  - 3. Time is divided into slots and nodes are synchronized on slot boundaries.
- Algorithm
  - In each time slot, a node independently transmits with probability $p_t$, a parameter to be determined, and listens with probability $1-p_t$.
  - A discovery is made if exactly one node transmits in a slot; otherwise no discovery is made in that time slot.

- Analysis
  - Consider a clique of n nodes numbered 1, . . . , n. The probability that node i successfully transmits in a given time slot is given by

$$p_s = p_t(1 - p_t)^{n-1}$$

  - The process of neighbor discovery can be treated as a coupon collector problem
    - A node C is regarded as a coupon collector
    - In each slot, C draws one of the n coupons (i.e. discovers a given node) with probability $p_s$, and draws no coupon (corresponding to an idle slot or a collision) with probability $1 - p_s$.
    - When C collects n distinct coupons, it means that it has discovered all of its n − 1 neighbors.
- Objective:
  - Choose $p_t$ so as to maximize the expected fraction of neighbors discovered in a given time slot

# Homework

- Paper reading
  - Sudarshan Vasudevan et. al., Neighbor Discovery in Wireless Networks and the Coupon Collector's Problem, Mobicom 2009
  - Piyush Gupta  and P. R. Kumar, The Capacity of Wireless Networks, IEEE transactions on information theory, 2000